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Almost subadditive weight functions form Braun–Meise–Taylor theory of ultradistributions

Alexander V. Abanin [a,*], Pham Trong Tien [b]

[a] Southern Institute of Mathematics, Markus st. 22, Vladikavkaz 362027, Russia
[b] Southern Federal University, Milchakov st. 8a, Rostov-on-Don 344090, Russia

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As it is known, Roumieu–Komatsu theory of ultradistributions is strictly larger than Beurling–Björck one and that the latter theory is established by the class of all subadditive weight functions. In its own turn, Roumieu–Komatsu theory is equivalent to Braun–Meise–Taylor one which is given by the class of all weight functions. We prove that the class of all almost subadditive weight functions forms Braun–Meise–Taylor theory of ultradistributions.

1. In order to enlarge the class of L. Schwartz's distributions, many authors developed several ultradistributions theories. We only mention A. Beurling [4], G. Björck [5], R. Braun, R. Meise and B.A. Taylor [6], J. Ciorănescu and L. Zsidó [7], H. Komatsu [10], J.L. Lions and E. Magenes [11], C. Roumieu [13,14]. Each ultradistribution theory is based on a family $(\mathcal{D}_\alpha)_{\alpha\in A}$ consisting of spaces of test functions and satisfying some formal assumptions (see [7, Section 7]). If $(\mathcal{D}_\alpha)_{\alpha\in A}$ and $(\mathcal{D}_\beta)_{\beta\in B}$ are theories of ultradistributions, then $(\mathcal{D}_\alpha)_{\alpha\in A}$ is called larger than $(\mathcal{D}_\beta)_{\beta\in B}$ if for each $\beta \in B$ there exists $\alpha \in A$ such that $\mathcal{D}_\alpha \subset \mathcal{D}_\beta$. The two theories are equivalent if each one is larger than the other.

By [7] Roumieu–Komatsu theory is strictly larger than Beurling–Björck one while Ciorănescu–Zsidó and Roumieu–Komatsu theories are equivalent in one-dimensional case. In [6] it has been shown that Braun–Meise–Taylor theory is equivalent to Roumieu–Komatsu one in N-dimensional case and extends Ciorănescu–Zsidó theory to $\mathbb{R}^N$. In this connection we remind that Beurling–Björck and Braun–Meise–Taylor theories are defined by systems of weight functions. A *weight function*, in the sense of [6], is a continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(α) $\omega(2t) = O(\omega(t))$ as $t \to \infty$;
(β) $\int_1^\infty \frac{\omega(t)}{t^2}\,dt < \infty$;
(γ) $\log t = o(\omega(t))$ as $t \to \infty$;
(δ) $\varphi_\omega(x) := \omega(e^x)$ is convex.

Denote by Ω the set of all weight functions and define for $\omega \in \Omega$ the space

$$\mathcal{D}_{(\omega)} := \left\{ f \in \mathcal{D}(\mathbb{R}^N) \colon \int\limits_{\mathbb{R}^N} \left|\hat{f}(x)\right| e^{n\omega(|x|)}\,dx < \infty \text{ for all } n \in \mathbb{N} \right\},$$

* Corresponding author.
E-mail address: abanin@math.rsu.ru (A.V. Abanin).

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where $\mathcal{D}(\mathbb{R}^N)$ is the space of all infinitely differentiable functions with compact support in $\mathbb{R}^N$ and $\hat{f}$ is the Fourier transformation of f. Then $(\mathcal{D}_{(\omega)})_{\omega\in\Omega}$ constitutes Braun–Meise–Taylor theory.

It is useful to have subclasses $\Sigma \subset \Omega$ for which $(\mathcal{D}_{(\sigma)})_{\sigma\in\Sigma}$ forms the ultradistribution theory equivalent to $(\mathcal{D}_{(\omega)})_{\omega\in\Omega}$. Such Σ we will call a *sufficient subclass of weight functions* or simply *sufficient*. As it is known (see [5, Theorem 3.1.8], and [8]), $\mathcal{D}_{(\sigma)} \subset \mathcal{D}_{(\omega)}$ if and only if ω is dominated by σ, i.e. if there exists a positive constant C such that $\omega(t) \leqslant C\sigma(t)$ for all t large enough. This implies that $\Sigma \subset \Omega$ is a sufficient subclass if and only if each $\omega \in \Omega$ is dominated by some $\sigma \in \Sigma$. Put $S := \{\sigma \in \Omega\colon\ \sigma \text{ is subadditive}\}$ and note that by [5, Theorem 1.2.7], and [7, 7.3], $(\mathcal{D}_{(\sigma)})_{\sigma\in S}$ forms Beurling–Björck theory. Since $(\mathcal{D}_{(\omega)})_{\omega\in\Omega}$ is strictly larger than $(\mathcal{D}_{(\sigma)})_{\sigma\in S}$, we have that S is not sufficient. In this connection we mention U. Franken [9, Proposition 3], who constructed an explicit weight function which cannot be dominated by any function $\sigma \in S$. In the present paper we prove that the class AS of all almost subadditive weight functions is sufficient and consequently $(\mathcal{D}_{(\nu)})_{\nu\in AS}$ gives Braun–Meise–Taylor theory of ultradistributions.

Definition 1.1. (See [2].) A function $f : [0,\infty) \to [0,\infty)$ is called almost subadditive if for each $p > 1$ there exists $C > 0$ such that

$$f(x+y) \leqslant p\big(f(x) + f(y)\big) + C \quad \text{for all } x, y \geqslant 0.$$

Remark 1.2. By [3] (see also [1, Chapter 1]) there exists an almost subadditive weight function which cannot be dominated by any subadditive weight function. Clearly, this is a refinement of the similar above-mentioned result of U. Franken.

Proposition 1.3. *For each weight function ω there exists an almost subadditive weight function ν such that $\omega(t) \leqslant \nu(t)$ for all $t \geqslant 0$.*

Proof. *Step 1*. Take a sequence $(\varepsilon_n)_{n=0}^{\infty}$ with $\varepsilon_0 = 1$ and $\varepsilon_n \downarrow 0$. For each $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ put

$$\sigma_n(t) := (1+\varepsilon_n)\int_1^{\infty} \frac{\omega(ts)}{s^{2+\varepsilon_n}}\,ds = (1+\varepsilon_n)t^{1+\varepsilon_n}\int_t^{\infty} \frac{\omega(s)}{s^{2+\varepsilon_n}}\,ds, \quad t \geqslant 0. \tag{1.1}$$

Obviously, σ_n is continuous on $[0,\infty)$. Since ω is increasing on $[0,\infty)$, the same is σ_n, and

$$\sigma_n(t) \geqslant (1+\varepsilon_n)\int_1^{\infty} \frac{\omega(t)}{s^{2+\varepsilon_n}}\,ds = \omega(t) \quad \text{for all } t \geqslant 0. \tag{1.2}$$

Next, note that condition (β) implies that $\omega(t) = o(t)$ as $t \to \infty$ (see [12, 1.2(b)]). Then, integrating by parts, we find

$$\sigma_n(t) = -\int_1^{\infty} \omega(ts)\,d\left(\frac{1}{s^{1+\varepsilon_n}}\right) = \omega(t) + \int_1^{\infty} \frac{d(\omega(ts))}{s^{1+\varepsilon_n}}.$$

Since ω is increasing, we then have that $\sigma_n(t) \leqslant \sigma_{n+1}(t)$ for all $t \geqslant 0$.

Step 2. Construct the function σ as follows. Choose $t_1 > 1$ so that

$$\int_{t_1}^{\infty} \frac{\omega(s)}{s^2}\,ds < \frac{\varepsilon_1}{1+\varepsilon_1}$$

and set

$$y_0 := \frac{\sigma_0(t_1)}{t_1^{1+\varepsilon_0}} = (1+\varepsilon_0)\int_{t_1}^{\infty} \frac{\omega(s)}{s^{2+\varepsilon_0}}\,ds.$$

Let $u_1 \geqslant t_1$ be a unique solution of the equation $y_0 t^{1+\varepsilon_0} = \sigma_1(t)$. To prove the existence of u_1, observe that

$$\frac{\sigma_1(t)}{t^{1+\varepsilon_0}} = (1+\varepsilon_1)t^{\varepsilon_1-\varepsilon_0}\int_t^{\infty} \frac{\omega(s)}{s^{2+\varepsilon_1}}\,ds.$$

The function on the right is continuous on $[1,\infty)$, strictly decreases to 0 as $t \uparrow \infty$, and, by step 1,

$$\frac{\sigma_1(t_1)}{t_1^{1+\varepsilon_0}} \geqslant \frac{\sigma_0(t_1)}{t_1^{1+\varepsilon_0}} = y_0.$$

Thus, the considered equation has a solution on $[t_1,\infty)$.

Put

$$\sigma(t) := \begin{cases} \sigma_0(t), & t \in [0, t_1], \\ y_0 t^{1+\varepsilon_0}, & t \in (t_1, u_1) \end{cases}$$

and suppose that $\sigma(t)$ is already defined on $[0, u_n)$ for some $n \in \mathbb{N}$. Then we set

$$\sigma(t) := \begin{cases} \sigma_n(t), & t \in [u_n, t_{n+1}], \\ y_n t^{1+\varepsilon_n}, & t \in (t_{n+1}, u_{n+1}), \end{cases}$$

where:

1. $t_{n+1} > \max\{t_n, n+1\}$ is taken so that
$$\int_{t_{n+1}}^{\infty} \frac{\omega(s)}{s^2}\, ds < \frac{\varepsilon_{n+1}}{(1+\varepsilon_{n+1})(n+1)^2}; \tag{1.3}$$
2. $y_n := \frac{\sigma_n(t_{n+1})}{t_{n+1}^{1+\varepsilon_n}} = (1+\varepsilon_n)\int_{t_{n+1}}^{\infty} \frac{\omega(s)}{s^{2+\varepsilon_n}}\, ds$;
3. $u_{n+1} \geqslant t_{n+1}$ is a unique solution of the equation $y_n t^{1+\varepsilon_n} = \sigma_{n+1}(t)$, i.e. $y_n u_{n+1}^{1+\varepsilon_n} = \sigma_{n+1}(u_{n+1})$ (the existence of u_{n+1} is checked as above).

Thus, σ is well defined on $[0, \infty)$.

Step 3. We claim that $\omega(t) \leqslant \sigma(t)$ for all $t \geqslant 0$ and σ satisfies all properties of a weight function except, may be, (α) and (δ).

Indeed, by the choice of y_n,

$$y_n t^{1+\varepsilon_n} = (1+\varepsilon_n) t^{1+\varepsilon_n} \int_{t_{n+1}}^{\infty} \frac{\omega(s)}{s^{2+\varepsilon_n}}\, ds \geqslant \sigma_n(t) \quad \text{for all } t \geqslant t_{n+1}.$$

Therefore $\sigma(t) \geqslant \sigma_n(t)$ for all $t \in [u_n, u_{n+1})$. Using (1.2), we then find that $\sigma(t) \geqslant \omega(t)$ for all $t \geqslant 0$. From this it follows immediately that σ satisfies (γ).

Next, by the choice of y_n and u_n it is clear that σ is continuous and increasing on $[0, \infty)$.

Now, show that σ has property (β) of a weight function. Using the choice of y_n and u_{n+1} again, we have that

$$\sigma(t) = y_n t^{1+\varepsilon_n} \leqslant \sigma_{n+1}(t) \quad \text{for all } t \in (t_{n+1}, u_{n+1}).$$

Hence, $\sigma(t) \leqslant \sigma_n(t)$ for all $t \in [t_n, t_{n+1})$ and $n \in \mathbb{N}$. Applying (1.3), observe that

$$\int_{t_n}^{t_{n+1}} \frac{\sigma_n(t)}{t^2}\, dt \leqslant (1+\varepsilon_n) \int_{t_n}^{\infty} t^{\varepsilon_n - 1} \int_{t}^{\infty} \frac{\omega(s)}{s^{2+\varepsilon_n}}\, ds\, dt$$
$$= \frac{1+\varepsilon_n}{\varepsilon_n} \int_{t_n}^{\infty} \omega(s) \frac{s^{\varepsilon_n} - t_n^{\varepsilon_n}}{s^{2+\varepsilon_n}}\, ds \leqslant \frac{1+\varepsilon_n}{\varepsilon_n} \int_{t_n}^{\infty} \frac{\omega(s)}{s^2}\, ds \leqslant \frac{1}{n^2}.$$

Consequently,

$$\int_{t_1}^{\infty} \frac{\sigma(t)}{t^2}\, dt \leqslant \sum_{n=1}^{\infty} \int_{t_n}^{t_{n+1}} \frac{\sigma_n(t)}{t^2}\, dt \leqslant \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

that is σ satisfies (β).

So, σ has all properties of a weight function except, may be, (α) and (δ) and $\omega(t) \leqslant \sigma(t)$ for all $t \geqslant 0$ (it can be shown that σ has (α) but we need not use this property in our further considerations).

Step 4. Define

$$\mu|_{[0,1]} := 0, \qquad \mu(t) := \int_1^t \frac{\sigma(s)}{s}\, ds \quad \text{for } t \geqslant 1,$$

and prove that μ is an almost subadditive weight function. Obviously, μ is an increasing continuous function on $[0,\infty)$ and $\mu \in C^1([1,\infty))$. Observe that

$$\varphi_\mu(x) = \mu(e^x) = \int_1^{e^x} \frac{\sigma(s)}{s}\,ds = \int_0^x \varphi_\sigma(t)\,dt, \quad x \geqslant 0.$$

Since $\varphi'_\mu(x) = \varphi_\sigma(x)$ increases, φ_μ is convex on $[0,\infty)$.

Next,

$$\sigma(t) \leqslant \int_t^{et} \frac{\sigma(s)}{s}\,ds \leqslant \mu(et) = \mu(t) + \int_t^{et} \frac{\sigma(s)}{s}\,ds \leqslant \mu(t) + \sigma(et) \quad \text{for all } t \geqslant 1. \tag{1.4}$$

Since σ satisfies (γ), from this it follows that μ has (γ) too. Further, note that

$$\int_1^\infty \frac{\mu(t)}{t^2}\,dt = \int_1^\infty \frac{1}{t^2}\int_1^t \frac{\sigma(s)}{s}\,ds\,dt = \int_1^\infty \frac{\sigma(s)}{s}\,ds \int_s^\infty \frac{dt}{t^2} = \int_1^\infty \frac{\sigma(s)}{s^2}\,ds < \infty.$$

Thus, μ satisfies (β).

Now prove that μ is an almost subadditive function. For each $n \in \mathbb{N}$ put

$$C_n := \sigma(1) + \sum_{k=0}^{n-1}(\varepsilon_k - \varepsilon_n)\int_{u_k}^{u_{k+1}} \frac{\sigma(s)}{s}\,ds, \quad \text{where } u_0 := 1,$$

and consider the function

$$f(t) := t^{-1-\varepsilon_n}\big(\mu(t) + C_n\big), \quad t \geqslant 1.$$

Clearly, f is a C^1-function on $[1,\infty)$ and, by direct calculations,

$$f'(t) := t^{-2-\varepsilon_n}\big(\sigma(t) - (1+\varepsilon_n)\mu(t) - (1+\varepsilon_n)C_n\big), \quad t \geqslant 1. \tag{1.5}$$

Using (1.1), find that, for $t \in (u_n, t_{n+1})$,

$$\begin{aligned}\sigma'(t) &= (1+\varepsilon_n)\left((1+\varepsilon_n)t^{\varepsilon_n}\int_t^\infty \frac{\omega(s)}{s^{2+\varepsilon_n}}\,ds - \frac{\omega(t)}{t}\right)\\ &= (1+\varepsilon_n)\frac{\sigma(t)-\omega(t)}{t} \leqslant (1+\varepsilon_n)\frac{\sigma(t)}{t}.\end{aligned}$$

Next, for $t \in [t_{n+1}, u_{n+1}]$,

$$\sigma'(t) = y_n(1+\varepsilon_n)t^{\varepsilon_n} = (1+\varepsilon_n)\frac{\sigma(t)}{t}.$$

Since $(\varepsilon_n)_{n=1}^\infty$ is decreasing, we then have that

$$\sigma'(t) \leqslant (1+\varepsilon_n)\frac{\sigma(t)}{t} \quad \text{for all } t \geqslant u_n.$$

Therefore, for $t \geqslant u_n$,

$$\begin{aligned}\sigma(t) &= \sigma(1) + \sum_{k=0}^{n-1}\int_{u_k}^{u_{k+1}} \sigma'(s)\,ds + \int_{u_n}^t \sigma'(s)\,ds\\ &\leqslant \sigma(1) + \sum_{k=0}^{n-1}(1+\varepsilon_k)\int_{u_k}^{u_{k+1}} \frac{\sigma(s)}{s}\,ds + (1+\varepsilon_n)\int_{u_n}^t \frac{\sigma(s)}{s}\,ds\\ &= \sigma(1) + (1+\varepsilon_n)\int_1^t \frac{\sigma(s)}{s}\,ds + \sum_{k=0}^{n-1}(\varepsilon_k-\varepsilon_n)\int_{u_k}^{u_{k+1}} \frac{\sigma(s)}{s}\,ds = (1+\varepsilon_n)\mu(t) + C_n.\end{aligned}$$

Using (1.5), we then have that

$$f'(t) \leqslant -\varepsilon_n C_n t^{-2-\varepsilon_n} < 0 \quad \text{for all } t \geqslant u_n \text{ and } n \in \mathbb{N}_0.$$

Hence, f decreases on $[u_n, \infty)$.

Let $y \geqslant x \geqslant u_n$. Applying the fact that f is decreasing, we have

$$\frac{\mu(x+y)+C_n}{(x+y)^{1+\varepsilon_n}} \leqslant \frac{\mu(x)+C_n}{x^{1+\varepsilon_n}} \quad \text{and} \quad \frac{\mu(x+y)+C_n}{(x+y)^{1+\varepsilon_n}} \leqslant \frac{\mu(y)+C_n}{y^{1+\varepsilon_n}},$$

and therefore,

$$\bigl(x^{1+\varepsilon_n} + y^{1+\varepsilon_n}\bigr)\bigl(\mu(x+y)+C_n\bigr) \leqslant (x+y)^{1+\varepsilon_n}\bigl(\mu(x)+\mu(y)+2C_n\bigr).$$

Note that $(x+y)^{1+\varepsilon_n} \leqslant 2^{\varepsilon_n}(x^{1+\varepsilon_n} + y^{1+\varepsilon_n})$. Thus,

$$\mu(x+y)+C_n \leqslant 2^{\varepsilon_n}\bigl(\mu(x)+\mu(y)+2C_n\bigr).$$

From this it easily follows that μ is an almost subadditive function. In particular, μ satisfies condition (α) of a weight function.

Taking into account all facts obtained, we have that μ is an almost subadditive weight function. Using (1.4), property (α) for μ, and inequality $\omega(t) \leqslant \sigma(t)$, we have that there exists $M > 0$ so that

$$\omega(t) \leqslant \sigma(t) \leqslant M\bigl(\mu(t)+1\bigr) \quad \text{for all } t \geqslant 0.$$

From this it follows that $\nu(t) := M(\mu(t)+1)$ is an almost subadditive weight function with $\omega(t) \leqslant \nu(t)$ for all $t \geqslant 0$. This completes the proof. $\square$

Theorem 1.4. *The class AS of all almost subadditive weight functions is a sufficient subclass of weight functions.*

Proof. Direct consequence of Proposition 1.3. $\square$

Remark 1.5 *(Concluding remarks).* Two weight functions, ω and σ, are called equivalent if each one is dominated by the other. Remind (see [12]) that a weight function ω is called strong if there exists $C > 0$ such that

$$\int_1^\infty \frac{\omega(ty)}{t^2}\,dt \leqslant C\bigl(\omega(y)+1\bigr) \quad \text{for all } y \geqslant 0.$$

By [12] for each strong weight function ω there exists an equivalent concave weight function ν. Obviously, ν is subadditive. Thus, we have the following refinement of Proposition 1.3 for strong weight functions:

For each strong weight function there exists an equivalent subadditive weight function.

By the above-mentioned result of U. Franken this statement is not valid for an arbitrary weight function. Thus, it is interesting to know whether or not for each weight function there exists an equivalent almost subadditive weight function. If the answer is positive, then $(\mathcal{D}_{(\nu)})_{\nu\in AS} = (\mathcal{D}_{(\omega)})_{\omega\in\Omega}$, while if not, then by Theorem 1.4 $(\mathcal{D}_{(\nu)})_{\nu\in AS}$ is a proper subfamily of $(\mathcal{D}_{(\omega)})_{\omega\in\Omega}$ which forms Braun–Meise–Taylor theory of ultradistributions. In this connection we note that the almost subadditive weight function ν constructed in the proof of Proposition 1.3 might not be equivalent to ω. Indeed, let $\omega(t) = \ln^2(1+t)$. Using notation of the proof of Proposition 1.3 and inequality $\omega(t) \leqslant \sigma(t)$, we have that

$$\begin{aligned}\limsup_{t\to\infty} \frac{\omega(t)}{\nu(t)} &= \frac{1}{M}\limsup_{t\to\infty}\frac{\omega(t)}{\mu(t)} \leqslant \frac{1}{M}\limsup_{t\to\infty}\frac{\omega'(t)}{\mu'(t)} \\ &= \frac{1}{M}\limsup_{t\to\infty}\frac{t\omega'(t)}{\sigma(t)} \leqslant \frac{1}{M}\limsup_{t\to\infty}\frac{t\omega'(t)}{\omega(t)} = 0.\end{aligned}$$

Hence, the function ω is dominated by ν but it is not equivalent to ν.

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